Extending Wang's 1D Sn Analytic Solution to Heterogeneous Problems with No Iterations on Interfacial Fluxes

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Introduction

- Analytic techniques for one-dimensional (1D) transport problems can produce extremely accurate benchmark solutions with no spatial errors.
- Analytic transport solution provides an efficient means to verify new proposed numerical spatial discretization approaches for transport method development.
- Highly accurate 1D solutions can benefit the development of nodal transport methods which use the transverse integration approach to convert the multi-dimensional transport equation to a set of coupled 1D ones [1].
- Analytic solution for the 1D monoenergetic Sn neutron transport equation in slab geometry, has been frequently visited by many researchers over the last few decades [1-5].
- Most recently, Wang et al. employs an *eigen-decomposition* procedure of the transport-scattering operator in the Sn equation and yields a closed form of analytic solution [6]. This idea may bear some similarities to some former work [7, 8], and has the same essence as the analytic approach recently proposed by English and Wu [9].

1D Monoenergetic Sn Transport Equation

- With standard notations, the monoenergetic Sn transport equation in slab geometry with homogeneous media and constant external neutron source is written as

$$\mu_m \frac{d\psi_m(x)}{dx} + \Sigma_t(x)\psi_m(x) = \sum_{l=0}^{L} \frac{2l+1}{2} \Sigma_{sl}(x) P_l(\mu_m) \phi_l(x) + \frac{Q}{2},$$

where $m = 1, \cdots, N.$

- $\phi_l(x)$ is the angular flux moment given by

$$\phi_l(x) = \sum_{m'=1}^{N} w_{m'} P_l(\mu_{m'}) \psi_{m'}(x)$$

1D Sn Equation – Matrix-Vector Form

- The Sn equation can be written into a matrix-vector form

$$\frac{d\boldsymbol{\psi}(x)}{dx} + \mathbf{A}\boldsymbol{\psi}(x) = \mathbf{b}$$

where $\boldsymbol{\psi}(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_N(x) \end{bmatrix}, \quad \mathbf{b} = \frac{Q}{2}\begin{bmatrix} 1/\mu_1 \\ 1/\mu_2 \\ \vdots \\ 1/\mu_N \end{bmatrix}$

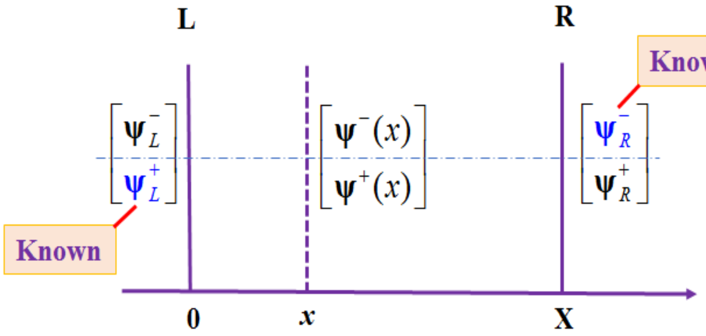
$$\mathbf{A} = \begin{bmatrix} \frac{1}{\mu_1}\left(\Sigma_t - \sum_{l=0}^{L}\frac{2l+1}{2}\Sigma_{sl}P_l(\mu_1)w_1P_l(\mu_1)\right) & -\frac{1}{\mu_1}\left(\sum_{l=0}^{L}\frac{2l+1}{2}\Sigma_{sl}P_l(\mu_1)w_2P_l(\mu_2)\right) & \cdots & -\frac{1}{\mu_1}\left(\sum_{l=0}^{L}\frac{2l+1}{2}\Sigma_{sl}P_l(\mu_1)w_NP_l(\mu_N)\right) \\ -\frac{1}{\mu_2}\left(\sum_{l=0}^{L}\frac{2l+1}{2}\Sigma_{sl}P_l(\mu_2)w_1P_l(\mu_1)\right) & \frac{1}{\mu_2}\left(\Sigma_t - \sum_{l=0}^{L}\frac{2l+1}{2}\Sigma_{sl}P_l(\mu_2)w_2P_l(\mu_2)\right) & \vdots & -\frac{1}{\mu_2}\left(\sum_{l=0}^{L}\frac{2l+1}{2}\Sigma_{sl}P_l(\mu_2)w_NP_l(\mu_N)\right) \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\mu_N}\left(\sum_{l=0}^{L}\frac{2l+1}{2}\Sigma_{sl}P_l(\mu_N)w_1P_l(\mu_1)\right) & -\frac{1}{\mu_N}\sum_{l=0}^{L}\frac{2l+1}{2}\Sigma_{sl}P_l(\mu_N)w_2P_l(\mu_2) & \cdots & \frac{1}{\mu_N}\left(\Sigma_t - \sum_{l=0}^{L}\frac{2l+1}{2}\Sigma_{sl}P_l(\mu_N)w_NP_l(\mu_N)\right) \end{bmatrix}$$

Boundary Conditions

- For a slab with a size of ***X*** cm, the left and right boundary conditions (<u>if non-reflective</u>) can be generally described as follows, respectively

$$\psi_m^L = \psi_m(0) = f_m, \qquad \textit{for} \;\; \mu_m > 0,$$
$$\psi_m^R = \psi_m(X) = g_m, \qquad \textit{for} \;\; \mu_m < 0.$$

where $\boldsymbol{f}_m$ and $\boldsymbol{g}_m$ are prescribed incident fluxes on the outer boundaries.



Eigen-Decomposition Procedure

- Suppose the matrix **A** is diagonalizable, then it can be decomposed with the standard eigen-decomposition procedure

$$\mathbf{A} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1}$$

- The original transport equation becomes

$$\frac{d\boldsymbol{\psi}(x)}{dx} + \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1}\boldsymbol{\psi}(x) = \mathbf{b}$$

- Define a ***pseudo-angular flux vector***

$$\boldsymbol{\varphi} = \mathbf{R}^{-1}\,\boldsymbol{\psi}$$

- The transport equation is re-written as

$$\frac{\partial\boldsymbol{\varphi}(x)}{\partial x} + \boldsymbol{\Lambda}\boldsymbol{\varphi}(x) = \mathbf{R}^{-1}\mathbf{b}$$

Wang's Closed Form Solution

$$\begin{bmatrix} \boldsymbol{\psi}^-(x) \\ \boldsymbol{\psi}^+(x) \end{bmatrix} = \mathbf{R} \begin{bmatrix} e^{\boldsymbol{\Lambda}^-(X-x)} & \\ & e^{-\boldsymbol{\Lambda}^+ x} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_R^- \\ \boldsymbol{\varphi}_L^+ \end{bmatrix} + \frac{Q}{2(\Sigma_t - \Sigma_{s0})} \left(\mathbf{I} - \mathbf{R} \begin{bmatrix} e^{\boldsymbol{\Lambda}^-(X-x)} & \\ & e^{-\boldsymbol{\Lambda}^+ x} \end{bmatrix} \mathbf{R}^{-1} \right) \mathbf{1}$$

where **I** is the identity matrix and **1** is a vector with all elements be one, the minus and plus superscripts in angular fluxes stand for fluxes with positive and negative ordinates in the Sn equation.

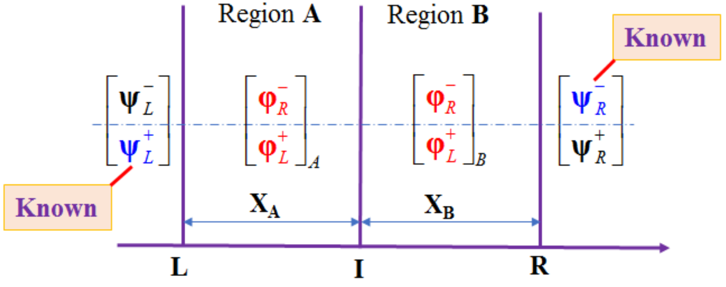
- The undetermined vector $\begin{bmatrix} \boldsymbol{\varphi}_R^- \\ \boldsymbol{\varphi}_L^+ \end{bmatrix}$ appearing in the solution are <u>the formal boundary values of the pseudo-fluxes</u>. This vector, which can be determined by physical boundary conditions, makes the closed form solution interesting because it is the only unknown in the solution and the two elements contained in the vector appears at different physical location of the problem.

D. WANG and T. BYAMBAAKHUU, "A New Analytical S_N Solution in Slab Geometry", *Trans. Am. Nucl. Soc.*, **117**, (2017).

Extension of the 1D Analytic Solution

- Deal with heterogeneous conditions
- Eliminate iterations on interfacial Fluxes
- Incorporate reflective boundary conditions
- Include void region situations

Heterogeneous Conditions



For illustration, a two-region problem is shown in the left figure, where the unknowns (i.e., the pseudo-flux vector) for each region are indicated with red color.

Define: $\begin{bmatrix} \boldsymbol{\varphi}_R^- \\ \boldsymbol{\varphi}_L^+ \end{bmatrix}_A = \begin{bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\varphi}_R^- \\ \boldsymbol{\varphi}_L^+ \end{bmatrix}_B = \begin{bmatrix} \boldsymbol{\varphi}_3 \\ \boldsymbol{\varphi}_4 \end{bmatrix}.$

The analytic solution can be written into following two equations (for both regions)

$$\boldsymbol{\psi}^-(x) = \mathbf{R}_{11} e^{\boldsymbol{\Lambda}^-(X-x)} (\boldsymbol{\varphi}_R^- - \mathbf{q}^-) + \mathbf{R}_{12} e^{-\boldsymbol{\Lambda}^+ x} (\boldsymbol{\varphi}_L^+ - \mathbf{q}^+) + \frac{Q}{2(\Sigma_t - \Sigma_{s0})} \mathbf{1}$$

$$\boldsymbol{\psi}^+(x) = \mathbf{R}_{21} e^{\boldsymbol{\Lambda}^-(X-x)} (\boldsymbol{\varphi}_R^- - \mathbf{q}^-) + \mathbf{R}_{22} e^{-\boldsymbol{\Lambda}^+ x} (\boldsymbol{\varphi}_L^+ - \mathbf{q}^+) + \frac{Q}{2(\Sigma_t - \Sigma_{s0})} \mathbf{1}$$

$$\left(\mathbf{q} = \begin{bmatrix} \mathbf{q}^- \\ \mathbf{q}^+ \end{bmatrix} = \frac{Q}{2(\Sigma_t - \Sigma_{s0})} \mathbf{R}^{-1} \mathbf{1} \right)$$

Four Equations for the Two-Region Example

1. At the left boundary (***x = L***), if the incoming flux ($\boldsymbol{\varphi}_L^+$) is known, we have

$$\left(\mathbf{R}_{21}e^{\boldsymbol{\Lambda}^- X}\right)_A \boldsymbol{\varphi}_1 + \left(\mathbf{R}_{22}\right)_A \boldsymbol{\varphi}_2 = \left(\mathbf{R}_{21}e^{\boldsymbol{\Lambda}^- X}\mathbf{q}^- + \mathbf{R}_{22}\mathbf{q}^+ - \frac{Q}{2(\Sigma_t - \Sigma_{s0})}\mathbf{1}\right)_A + \boldsymbol{\psi}_L^+$$

2. At the right boundary (***x = R***), the equation can be developed similarly

$$\left(\mathbf{R}_{11}\right)_B \boldsymbol{\varphi}_3 + \left(\mathbf{R}_{12}e^{-\boldsymbol{\Lambda}^+ X}\right)_B \boldsymbol{\varphi}_4 = \left(\mathbf{R}_{11}\mathbf{q}^- + \mathbf{R}_{12}e^{-\boldsymbol{\Lambda}^+ X}\mathbf{q}^+ - \frac{Q}{2(\Sigma_t - \Sigma_{s0})}\mathbf{1}\right)_B + \boldsymbol{\psi}_R^-$$

3. At the interface of the problem (***x = I***), if we set ***x = X***$_A$ to the analytic solution at Region A, and ***x = 0*** to the analytic solution at Region B, we obtain the two sets of angular flux solution at the interface. Using the flux continuity condition at the interface, we obtain

$$\left(\mathbf{R}_{11}\right)_A \boldsymbol{\varphi}_1 + \left(\mathbf{R}_{12}e^{-\boldsymbol{\Lambda}^+ X}\right)_A \boldsymbol{\varphi}_2 - \left(\mathbf{R}_{11}e^{\boldsymbol{\Lambda}^- X}\right)_B \boldsymbol{\varphi}_3 - \left(\mathbf{R}_{12}\right)_B \boldsymbol{\varphi}_4 = \left(\mathbf{R}_{11}\mathbf{q}^- + \mathbf{R}_{12}e^{-\boldsymbol{\Lambda}^+ X}\mathbf{q}^+ - \frac{Q}{2(\Sigma_t - \Sigma_{s0})}\mathbf{1}\right)_A - \left(\mathbf{R}_{11}e^{\boldsymbol{\Lambda}^- X}\mathbf{q}^- + \mathbf{R}_{12}\mathbf{q}^+ - \frac{Q}{2(\Sigma_t - \Sigma_{s0})}\mathbf{1}\right)_B$$

$$\left(\mathbf{R}_{21}\right)_A \boldsymbol{\varphi}_1 + \left(\mathbf{R}_{22}e^{-\boldsymbol{\Lambda}^+ X}\right)_A \boldsymbol{\varphi}_2 - \left(\mathbf{R}_{21}e^{\boldsymbol{\Lambda}^- X}\right)_B \boldsymbol{\varphi}_3 - \left(\mathbf{R}_{22}\right)_B \boldsymbol{\varphi}_4 = \left(\mathbf{R}_{21}\mathbf{q}^- + \mathbf{R}_{22}e^{-\boldsymbol{\Lambda}^+ X}\mathbf{q}^+ - \frac{Q}{2(\Sigma_t - \Sigma_{s0})}\mathbf{1}\right)_A - \left(\mathbf{R}_{21}e^{\boldsymbol{\Lambda}^- X}\mathbf{q}^- + \mathbf{R}_{22}\mathbf{q}^+ - \frac{Q}{2(\Sigma_t - \Sigma_{s0})}\mathbf{1}\right)_B$$

Incorporate Reflective Boundaries

$\mathbf{T}$ is a mirror reflective matrix: $\boldsymbol{\psi}_L^- = \mathbf{T}\boldsymbol{\psi}_L^+$

1. **If the left boundary (*x = L*) is reflective, we have**

$$\left[(\mathbf{R}_{21}-\mathbf{TR}_{11})e^{\boldsymbol{\Lambda}^- X}\right]_A \boldsymbol{\varphi}_1 + (\mathbf{R}_{22}-\mathbf{TR}_{12})_A \boldsymbol{\varphi}_2 = \left[(\mathbf{R}_{21}-\mathbf{TR}_{11})e^{\boldsymbol{\Lambda}^- X}\mathbf{q}^- + (\mathbf{R}_{22}-\mathbf{TR}_{12})\mathbf{q}^+ + \frac{Q}{2(\Sigma_t-\Sigma_{s0})}(\mathbf{T}-\mathbf{I})\mathbf{1}\right]_A$$

2. **If the right boundary (*x = R*) is reflective, we have**

$$(\mathbf{R}_{11}-\mathbf{TR}_{21})_B \boldsymbol{\varphi}_3 + \left[(\mathbf{R}_{12}-\mathbf{TR}_{22})e^{-\boldsymbol{\Lambda}^+ X}\right]_B \boldsymbol{\varphi}_4 = \left[(\mathbf{R}_{11}-\mathbf{TR}_{21})\mathbf{q}^- + (\mathbf{R}_{12}-\mathbf{TR}_{22})e^{-\boldsymbol{\Lambda}^+ X}\mathbf{q}^+ + \frac{Q}{2(\Sigma_t-\Sigma_{s0})}(\mathbf{T}-\mathbf{I})\mathbf{1}\right]_B$$

3. **At the interface of the problem (*x = I*), if we set *x = X_A* to the analytic solution at Region A, and *x = 0* to the analytic solution at Region B, we obtain the two sets of angular flux solution at the interface. Using the flux continuity condition at the interface, we obtain**

$$(\mathbf{R}_{11})_A \boldsymbol{\varphi}_1 + \left(\mathbf{R}_{12}e^{-\boldsymbol{\Lambda}^+ X}\right)_A \boldsymbol{\varphi}_2 - \left(\mathbf{R}_{11}e^{\boldsymbol{\Lambda}^- X}\right)_B \boldsymbol{\varphi}_3 - (\mathbf{R}_{12})_B \boldsymbol{\varphi}_4 = \left(\mathbf{R}_{11}\mathbf{q}^- + \mathbf{R}_{12}e^{-\boldsymbol{\Lambda}^+ X}\mathbf{q}^+ - \frac{Q}{2(\Sigma_t-\Sigma_{s0})}\mathbf{1}\right)_A - \left(\mathbf{R}_{11}e^{\boldsymbol{\Lambda}^- X}\mathbf{q}^- + \mathbf{R}_{12}\mathbf{q}^+ - \frac{Q}{2(\Sigma_t-\Sigma_{s0})}\mathbf{1}\right)_B$$

$$(\mathbf{R}_{21})_A \boldsymbol{\varphi}_1 + \left(\mathbf{R}_{22}e^{-\boldsymbol{\Lambda}^+ X}\right)_A \boldsymbol{\varphi}_2 - \left(\mathbf{R}_{21}e^{\boldsymbol{\Lambda}^- X}\right)_B \boldsymbol{\varphi}_3 - (\mathbf{R}_{22})_B \boldsymbol{\varphi}_4 = \left(\mathbf{R}_{21}\mathbf{q}^- + \mathbf{R}_{22}e^{-\boldsymbol{\Lambda}^+ X}\mathbf{q}^+ - \frac{Q}{2(\Sigma_t-\Sigma_{s0})}\mathbf{1}\right)_A - \left(\mathbf{R}_{21}e^{\boldsymbol{\Lambda}^- X}\mathbf{q}^- + \mathbf{R}_{22}\mathbf{q}^+ - \frac{Q}{2(\Sigma_t-\Sigma_{s0})}\mathbf{1}\right)_B$$

Account for Void Regions

- For a void region, the analytic solution is essentially reduced to the following form

$$\begin{bmatrix} \boldsymbol{\psi}^{-}(x) \\ \boldsymbol{\psi}^{+}(x) \end{bmatrix} = \mathbf{R} \begin{bmatrix} \boldsymbol{\varphi}_R^{-} \\ \boldsymbol{\varphi}_L^{+} \end{bmatrix}$$

where the eigenvector matrix ***R*** is really degraded to **an identity matrix**, which indicates the flux stays unchanged and the pseudo-flux becomes identical to the real flux.

Numerical Example 1

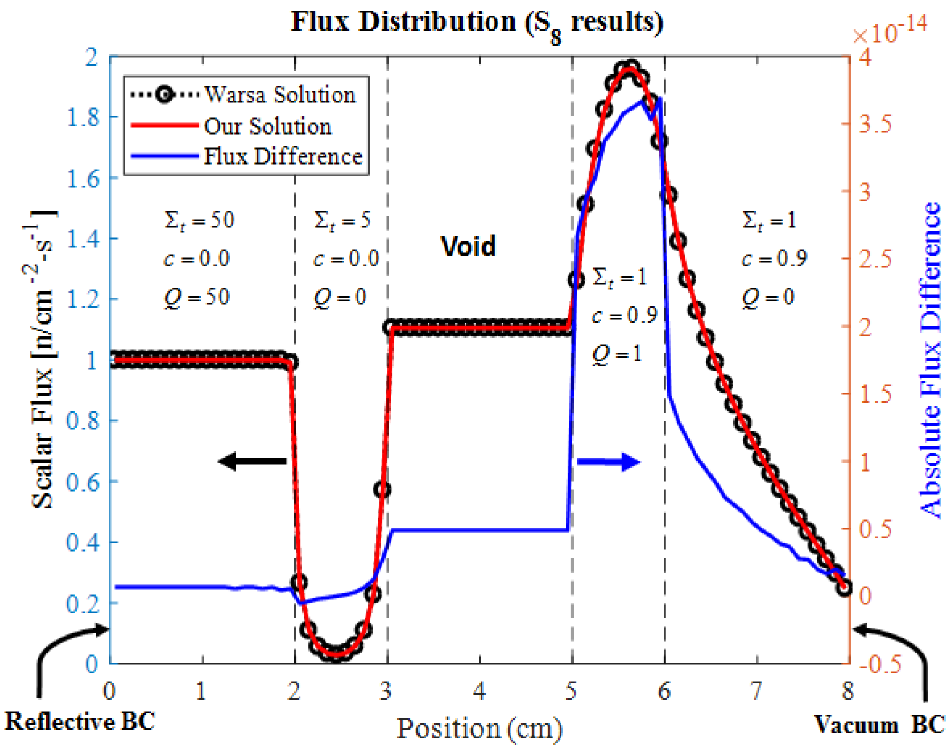
- Reed's problem **[Ref.]**

	Region 1	Region 2	Region 3	Region 4	Region 5
Σ_t [cm^{-1}]	50	5	0	1	1
Σ_s [cm^{-1}]	0	0	0	0.9	0.9
Q [scale]	50	0	0	1	0
x [cm]	$0 \leq x < 2$	$2 \leq x < 3$	$3 \leq x < 5$	$5 \leq x < 6$	$6 \leq x \leq 8$

- Reflective boundary on the left and vacuum boundary on the right side.

W.H. REED, "New difference schemes for the neutron transport equation," *Nucl. Sci. Eng.* **46**, 309 (1971).

Numerical Example 1 - Result



J. S. WARSA, "Analytical Sn Solutions in Heterogeneous Slabs Using Symbolic Algebra Computer Programs" *Annals of Nuclear Energy*, **29**, 851 (2002).

Numerical Example 2

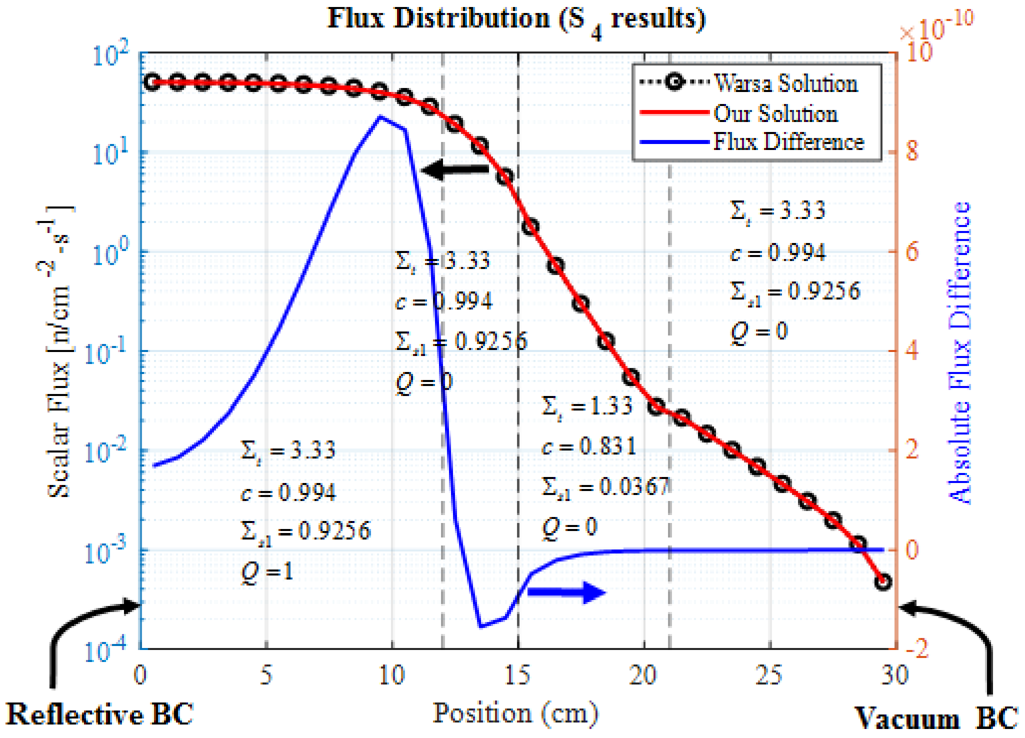
- Iron-water problem **[Ref.]**

	Region 1	Region 2	Region 3	Region 4
Σ_t [cm^{-1}]	3.33	3.33	1.33	3.33
c	0.994	0.994	0.831	0.994
Σ_{s1} [cm^{-1}]	0.9256	0.9256	0.0367	0.9256
Q [scale]	1	0	0	1
x [cm]	$0 \leq x < 2$	$2 \leq x < 3$	$3 \leq x < 5$	$5 \leq x < 6$

- Reflective boundary on the left and vacuum boundary on the right side.

E.W. Larsen, "On numerical solutions of transport problems in the diffusion limit," *Nucl. Sci. Eng.* **83**, 90 (1983).

Numerical Example 2 - Result



J. S. WARSA, "Analytical Sn Solutions in Heterogeneous Slabs Using Symbolic Algebra Computer Programs" *Annals of Nuclear Energy*, **29**, 851 (2002).

Future Perspectives

- Incorporate distributed source (i.e., the region source has spatial dependency)
- Cylindrical or spherical geometry
- More efficient way to solve for boundary and interfacial angular fluxes
- Extend to *k*-eigenvalue problem
- Etc.

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Questions?!

- Is matrix **A** always diagonalizable?
- Similarity and difference of this method comparing to **response matrix method** (RMM)?
- Computational cost and **efficiency** when approaching bigger multi-region problems?

Thanks for your time

Questions?